Synchronization of nonlinear oscillators: part II

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1 Synchronization: Definition
   - Perturbed nonlinear oscillators and coupled nonlinear oscillators
   - Locking

2 Phase equation
   - The reduction problem
   - Phase equation: derivation

3 Synchronization analysis
   - Synchronization of an oscillator with a periodic forcing
   - Synchronization of coupled oscillators
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Consider the ODE

$$\dot{x}(t) = f(x(t)) + \varepsilon g(x(t), t)$$

- $x : \mathbb{R}^+ \mapsto \mathbb{R}^n$ state vector
- $\varepsilon \in \mathbb{R}$ perturbation strength. Usually $\varepsilon \ll 1$
- $g : \mathbb{R}^n \times \mathbb{R}^+ \mapsto \mathbb{R}^n$ perturbation.

In absence of perturbation ($\varepsilon = 0$)

$$\begin{cases} 
\dot{x}_0(t) = f(x_0(t)) \\
x_0(t) = x_0(t + T)
\end{cases}$$
Coupled nonlinear oscillators

Consider the ODE

\[
\begin{align*}
\dot{x}_1(t) &= f_1(x_1(t)) + \varepsilon g_1(x_1(t), \ldots, x_N(t), t) \\
\vdots & \quad \vdots \\
\dot{x}_N(t) &= f_N(x_N(t)) + \varepsilon g_N(x_1(t), \ldots, x_N(t), t)
\end{align*}
\]

Alternative writings

\[
\dot{x}_i(t) = f_i(x_i(t)) + \varepsilon g_i(x_1(t), \ldots, x_N(t), t) \quad i = 1, \ldots, N
\]
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Synchronization: Definition
Phase equation
Synchronization analysis
Perturbed nonlinear oscillators and coupled nonlinear oscillators
Locking

Frequency locking

Consider two oscillators

\[ \dot{\phi}_1 = \omega_1 + \varepsilon h_1(\phi_1, \phi_2) \]
\[ \dot{\phi}_2 = \omega_2 + \varepsilon h_2(\phi_1, \phi_2) \]

The network is **frequency locked** when it has a stable periodic solution.

The limit cycle wraps on a \( T_2 \) torus

\[ T : [0, 2\pi] \times [0, 2\pi] \]
A periodic solution on the torus is called **torus knot**. If the first oscillator makes $m$ rotations while the second makes $n$ rotations they are $m : n$ **frequency locked**.

**Examples**

- 2 : 1 frequency locking
- 2 : 3 frequency locking
Phase locking

The two oscillators are \textbf{m : n phase locked} if

\[ m \phi_1(t) - n \phi_2(t) = \text{const} \]

Phase locking implies frequency locking, but not viceversa.
Synchronization

1 : 1 phase locking is called **Synchronization**

\[ \phi_1 - \phi_2 = 0 \rightarrow \text{In-phase synchronization} \]

\[ \phi_1 - \phi_2 = \pi \rightarrow \text{Anti-phase synchronization} \]
Synchronization visualization

Lissajous figures: plot the state variable of one oscillator vs the same variable of the other.

\[
\begin{align*}
    x_2 &= 0.5 x_1 \\
    x_2 &= -0.5 x_1
\end{align*}
\]
Synchronization visualization

Lissajous figures:
Lissajous figures:

An open curve fills the plane. No phase locking corresponds to quasi–periodic motion.
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In most practical situations we can derive the state equation of a given system from physical considerations. How can we transform to phase equation?

\[ \dot{x}(t) = f(x(t)) + \varepsilon g(x(t)) \quad \Rightarrow \quad \dot{\phi} = \omega + \varepsilon h(\phi, t) \]

- Malkin’s theorem
- Demir et al.
- Kuramoto
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Theorem (Malkin’s theorem, 1949)

Consider the perturbed oscillator

\[ \dot{x}(t) = f(x(t)) + \varepsilon g(x(t)) \]

such that the unperturbed equation \( \dot{x}(t) = f(x(t)) \) has an exponentially stable \( 2\pi \)-periodic solution \( x_0(t) \). Let \( \tau = \varepsilon t \) be the slow time and let \( \psi(\tau) \) be the phase deviation from the natural oscillation. Then the phase deviation is a solution to

\[ \frac{d\psi(\tau)}{d\tau} = \frac{1}{2\pi} \int_0^{2\pi} q^T(t + \psi) g(x_0(t + \psi)) \, dt \]

where \( q(t) \) is the unique non trivial solution of the adjoint problem

\[ \dot{q}(t) = -Df(x_0(t))^T q(t) \]

satisfying the normalization condition

\[ q(0)^T f(x_0(0)) = 1 \]
Sketch of the proof: Let \( x(t) = x_0(t + \psi(\tau)) + \varepsilon y(t) + \mathcal{O}(\varepsilon^2) \), then

\[
\dot{x}(t) = x_0'(t + \psi) \left(1 + \varepsilon \frac{d\psi}{d\tau}\right) + \varepsilon \dot{y}(t) + \ldots
\]

\[
= f(x_0(t + \psi) + y(t)) + \varepsilon g(x_0(t + \psi) + y(t)) + \ldots
\]

\[
= f(x_0(t + \psi)) + \varepsilon Df(x_0(t + \psi)) y(t) + \varepsilon g(x_0(t + \psi)) + \ldots
\]

Up to the first order in \( \varepsilon \)

\[
f(x_0(t + \psi)) \frac{d\psi}{d\tau} + \dot{y}(t) = Df(x_0(t + \psi)) y(t) + g(x_0(t + \psi))
\]

This is an equation of type

\[
\dot{y}(t) = Df(x_0(t + \psi)) y(t) + b(t + \psi)
\]

with

\[
b(t + \psi) = g(x_0(t + \psi)) - f(x_0(t + \psi)) \frac{d\psi}{d\tau}
\]
Using Fredholm alternative theorem the linear inhomogeneous system has a periodic (and therefore bounded) solution if and only if the orthogonality condition

$$\langle q, b \rangle = \int_{0}^{T} q^T(t + \psi) b(t + \psi) dt = 0$$

is satisfied for any solution $q(t)$ of the adjoint problem

$$\dot{q}(t) = -Df^T(x_0(t)) q(t)$$

The adjoint problem obeys the normalization condition

$$\int_{0}^{T} q^T(t) f(x_0(t)) dt = T$$

Using the definition of $b$ and the normalization condition

$$\frac{d\psi}{d\tau} = \frac{1}{T} \int_{0}^{T} q^T(t + \psi) g(x_0(t + \psi)) dt$$
Phase equation: Demir et al. derivation

- $g_1(x(t))$ tangent to the cycle
  
  $$g_1(x(t)) = u_1(t)v_1^T(t)g(x(t))$$

- $g_2(x(t))$ tangent to the isochron
  
  $$g_2(x(t)) = \sum_{k=2}^{n} u_k(t)v_k^T(t)g(x(t))$$
Theorem (Demir et al., 2000)

Consider the perturbed oscillator

$$\dot{x}(t) = f(x(t)) + \varepsilon g(x(t), t)$$

such that the unperturbed equation $\dot{x}(t) = f(x(t))$ has an exponentially stable $2\pi$–periodic solution $x_0(t)$. Let $\psi$ be the solution of the phase deviation

$$\dot{\psi} = \varepsilon v_1^T(t + \psi) g(x_0(t + \psi))$$

and let

$$g_1(x_0(t)) = u_1(t) v_1^T(t) g(x_0(t))$$

be the tangent perturbation. Then $x_0(t + \psi)$ solves

$$\dot{x}(t) = f(x(t)) + \varepsilon g_1(x(t), t)$$
Proof: Introducing the ansatz $x_0(t + \psi)$ as solution

\[
x'_0(t + \psi) \left(1 + \dot{\psi}\right) = f(x_0(t + \psi)) + \varepsilon g_1(x_0(t + \psi))
\]

\[
\dot{\psi} f(x_0(t + \psi)) = \varepsilon u_1(t + \psi) v_1^T(t + \psi) g(x_0(t + \psi))
\]

Remembering that $f(x_0(t)) = u_1(t)$ implies

\[
\dot{\psi} = \varepsilon v_1^T(t + \psi) g(x_0(t + \psi))
\]

that is true by hypothesis.
Theorem (Demir et al, 2000)

Consider the perturbed oscillator

\[ \dot{x}(t) = f(x(t)) + \varepsilon g(x(t), t) \]

such that the unperturbed equation \( \dot{x}(t) = f(x(t)) \) has an exponentially stable 2\( \pi \)-periodic solution \( x_0(t) \). Let \( \psi \) be the solution of the phase deviation

\[ \dot{\psi} = \varepsilon v_1^T (t + \psi) g(x_0(t + \psi)) \]

and let

\[ y(t) = \sum_{k=2}^{n} u_k(t + \psi) \int_{0}^{t+\psi} \exp(\mu_k(t + \psi - s)) v_k^T(s) g(x_0(s), s) \, ds \]

Then up to the first order in \( \varepsilon \) a solution for the perturbed oscillator is given by \( x(t) = x_0(t + \psi) + \varepsilon y(t) \).
**Phase equation: Demir et al. derivation, cont.**

*Sketch of the Proof:* Rewrite the system as

\[
\dot{x}(t) = f(x(t)) + \varepsilon (g_1(x(t), t) + g_2(x(t), t))
\]

Introducing the ansatz \(x(t) = x_0(t+\psi) + \varepsilon y(t)\) gives

\[
\dot{y}(t) = Df(x_0(t+\psi)) y(t) + g_2(x_0(t+\psi), t)
\]

Using variation of constants formula

\[
y(t) = \sum_{k=1}^{n} u_k(t+\psi) \int_0^{t+\psi} \exp(\mu_k(t+\psi-s)) v_k^T(s) g_2(x_0(s), s) ds
\]

that implies

\[
y(t) = \sum_{k=2}^{n} u_k(t+\psi) \int_0^{t+\psi} \exp(\mu_k(t+\psi-s)) v_k^T(s) g(x_0(s), s) ds
\]
Consider the perturbed oscillator

\[ \dot{x}(t) = f(x(t)) + \varepsilon g(x(t), t) \]

such that the unperturbed equation \( \dot{x}(t) = f(x(t)) \) has an exponentially stable \( T \)-periodic solution \( x_0(t) \). Let \( \phi(x) \) be the phase function. Then the phase function solves

\[ \dot{\phi} = \omega_0 + \varepsilon \nabla \phi(x_0(\phi)) \cdot g(x_0(\phi), t) \]
Proof: Let \( \phi(x) \) be the phase function defined through isochrons, then

\[
\dot{\phi} = \nabla \phi(x) \cdot \dot{x} = \nabla \phi(x) (f(x) + \varepsilon g(x, t)) = \omega_0 + \varepsilon \nabla \phi(x) \cdot g(x, t)
\]

We can invert \( \phi(x) \) to obtain \( x = x(\phi) \) at least in a small neighborhood of the cycle. Thus

\[
\dot{\phi} = \omega_0 + \varepsilon \nabla \phi(x(\phi)) \cdot g(x(\phi), t)
\]

Now let us make the ansatz \( x(\phi) = x_0(\phi) + \varepsilon y(\phi) \). Then up to the first order

\[
\dot{\phi} = \omega_0 + \varepsilon \nabla \phi(x_0(\phi)) \cdot g(x_0(\phi), t)
\]
Phase equation: considerations

- **Malkin**: \[ \frac{d\psi}{d\tau} = \frac{1}{2\pi} \int_0^{2\pi} q^T(t + \psi) g(x_0(t + \psi)) \, dt \]

- **Demir**: \[ \dot{\psi} = \varepsilon v_1^T(t + \psi) g(x_0(t + \psi), t) \]

- **Kuramoto**: \[ \dot{\phi} = \omega_0 + \varepsilon \nabla \phi(x_0(\phi)) \cdot g(x_0(\phi, t)) \]

**Theorem**

\((Kuramoto \Leftrightarrow Demir) \Rightarrow Malkin\)
Phase equation: considerations

**Kuramoto ⇒ Demir**
Introducing the phase deviation $\psi = \phi - \omega_0 t$ and for $\omega_0 = 1$
Kuramoto equation becomes

$$\dot{\psi} = \varepsilon \nabla \phi(x_0(t + \psi)) \cdot g(x_0(t + \psi), t)$$

The linear space tangent to the isochrons on the limit cycle is
spanned by $\{u_2(t), \ldots, u_n(t)\}$, thus $\nabla \phi(x_0(t)) = \lambda v_1^T(t)$ for some $\lambda \in \mathbb{R}$.
On the other hand by definition

$$\nabla \phi(x(t)) \cdot f(x(t)) = \omega_0 = 1$$

On the limit cycle

$$\lambda v_1^T(t) u_1(t) = 1 \Rightarrow \lambda = 1$$
Demir ⇒ Malkin

\[ \dot{\psi} = \varepsilon \mathbf{v}_1^T (t + \psi) \mathbf{g}(x_0(t + \psi), t) \]

For \( \varepsilon \to 0 \), \( \psi \) is a slow variable. We can average the right hand side over one period without introducing a great error (averaging)

\[ \frac{d\psi}{\varepsilon dt} = \frac{d\psi}{d\tau} = \frac{1}{2\pi} \int_0^{2\pi} \mathbf{v}_1^T (t + \psi) \mathbf{g}(x_0(t + \psi), t) \, dt \]
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Consider the problem

\[
\begin{align*}
\dot{x}(t) &= f(x(t)) + \varepsilon g(x(t), t) \\
g(x(t), t) &= g(x(t), t + T)
\end{align*}
\]

It can be reduced to the phase equation

\[\dot{\phi} = \omega_0 + \varepsilon \nabla \phi(x_0(\phi)) \cdot g(x_0(\phi), t)\]

Introduce the phase difference \(\psi = \phi - \omega t\) with \(\omega = 2\pi / T\) we get

\[\dot{\psi} = \Delta \omega + \varepsilon q(\omega_0 t + \psi, t)\]

- \(q(\omega t + \psi, t) = \nabla \phi(x_0(\omega t + \psi)) \cdot g(x_0(\omega t + \psi), t)\)
- \(\Delta \omega = \omega_0 - \omega\) is the frequency mismatch.
If the frequency mismatch is small we can average

$$\dot{\psi} = \Delta \omega + \varepsilon Q(\psi)$$

where $Q(\psi) = \frac{1}{T} \int_0^T q(\omega t + \psi) dt$

**Equilibrium points $\Rightarrow$ synchronized states**

$$\Delta \omega + \varepsilon Q(\psi) = 0$$

$Q(\psi)$ is a periodic function
Synchronization with a periodic forcing, cont.

- $\Delta \omega + \varepsilon Q(\psi) = 0$
- Synchronous states are alternatively stable and unstable
- As the frequency mismatch is increased (or decreased) synchronous states collide and vanish through fold bifurcations
Synchronization with a periodic forcing, cont.

\[ \Delta \omega + \varepsilon Q(\psi) = 0 \]

implies

\[ \varepsilon \min(Q(\psi)) \leq \Delta \omega \leq \varepsilon \max(Q(\psi)) \]

We can find synchronization regions

Arnold’s tongues

\[ \varepsilon \]

\[ \Delta \omega / \omega_0 \]

\[ 0.3 \quad 0.5 \quad 1 \]

\[ 1:1 \quad 1:2 \quad 1:3 \]
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Synchronization: Definition
Phase equation
Synchronization analysis

Synchronization of coupled oscillators

Coupled oscillators can be analyzed more or less the same way. The network

\[
\begin{align*}
\dot{x}_1(t) &= f_1(x_1(t)) + \varepsilon g_1(x(t), \ldots, x_N(t)) \\
& \quad \vdots \\
\dot{x}_N(t) &= f_N(x_1(t)) + \varepsilon g_N(x(t), \ldots, x_N(t))
\end{align*}
\]

can be reduced to the phase equation

\[
\begin{align*}
\dot{\phi}_1 &= \omega_1 + \varepsilon h_1(\phi_1, \ldots, \phi_N) \\
& \quad \vdots \\
\dot{\phi}_N &= \omega_N + \varepsilon h_N(\phi_1, \ldots, \phi_N)
\end{align*}
\]
Synchronizing the phase differences $\psi_i = \phi_i - \omega_0 t$ measured with respect to an "ideal" oscillator

\[
\begin{align*}
\dot{\psi}_1 &= \Delta \omega_1 + \varepsilon h_1(\omega_0 t + \psi_1, \ldots, \omega_0 t + \psi_N) \\
\vdots \\
\dot{\psi}_N &= \Delta \omega_N + \varepsilon h_N(\omega_0 t + \psi_1, \ldots, \omega_0 t + \psi_N)
\end{align*}
\]

- if the frequency mismatches $\Delta \omega_i = \omega_i - \omega_0$ are small, the equations can be averaged to remove the time dependency.
Synchronization of coupled oscillators, cont.

\[
\begin{align*}
\dot{\psi}_1 &= \Delta \omega_1 + \varepsilon H_1(\psi_1, \ldots, \psi_N) \\
\vdots & \quad \vdots \\
\dot{\psi}_N &= \Delta \omega_N + \varepsilon H_N(\psi_1, \ldots, \psi_N)
\end{align*}
\]

where

\[
H_i(\psi_1, \ldots, \psi_N) = \frac{1}{T} \int_0^T h_i(\omega_0 t + \psi_1, \ldots, \omega_0 t + \psi_N) \, dt
\]

Equilibrium points $\Rightarrow$ synchronized states

Stability analysis $\Rightarrow$ Eigenvalues
Essential bibliography